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Sampled-Data System Approach to Model Time-Division Switches

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The sampling switch in a time-division switching system is, in general, different from the sampler of sampled-data system theory. A general approach is developed for characterizing such a switch as an ideal sampler plus some modified transfer functions. With this approach, a time-division switching circuit containing a sampling switch can be converted easily to a typical sampled-data system, and the well-established mathematical tools for sampled-data systems, such as the Z-transform, can be applied. In addition, a simplified approach is described that will lead to a very good approximation of the "exact" solution.

I. INTRODUCTION

The transfer function approach developed for sampled-data systems has proved to be a very powerful tool for analyzing time-division systems.[1–3] It yields information useful for both analysis and synthesis of the system. However, its application is often limited due to the fact that the sampling switch in a time-division circuit is different from the sampler of sampled-data-system theory. This difference can be seen from the fact that the voltage at the output side of a sampler in a sampled-data system is always zero between sampling instants, while the voltage at the output side of a sampling switch in a time-division circuit is not necessarily zero between sampling instants, if, for example, the switch is connected to a capacitor. As a consequence, one cannot treat a time-division circuit as a sampled-data system unless the sampling switch can be modeled by a sampler plus a modified system-transfer function.

Of the few who have worked on time-division-system analysis,[1–9] only Desoer[1] has come close to using functional blocks to model a sampling switch, but no general approach has been developed. It is

the purpose of this paper to present a general approach for solving this problem. With this general approach, any time-division circuit containing a sampling switch can be converted to a typical sampled-data system, and the well-established mathematical tools for sampled-data systems, such as the Z-transform, can be applied.

II. FORMULATION

In a sampled-data system, the sampled signal is related to the original signal by a sampling device such as is shown in Fig. 1. The output of the samples is a train of amplitude-modulated pulses. The interval T between the consecutive pulses is called the sampling period, and the pulse width p is referred to as the sampling duration. In the ideal case, we assume that the sampler operates in zero time so that the pulse width p is equal to zero. Then the output of an ideal sampler is a train of amplitude-modulated impulses and is related to the input by

$$v^*(t) = \sum_{n=0}^{\infty} v(nT)\delta(t - nT), \tag{1}$$

where δ is the Dirac Delta function. We note that whether the operating time of the sampler is zero or not, the sampled voltage is always zero between samplings.

A switch operating periodically in a time-division system is not equivalent to a sampler in a sampled-data system, because the signal at the output side of a switch is not necessarily zero between samplings. However, if an ideal amplifier with zero output and infinite input impedances is added to the switch, as shown in Fig. 2, then the output signal of the amplifier is equal to zero between samplings. In fact, if the sampling duration is much shorter than the sampling period, then an input-output signal relation identical to (1) can be obtained. Thus, if the switch in a time-division circuit is followed by an amplifier, then

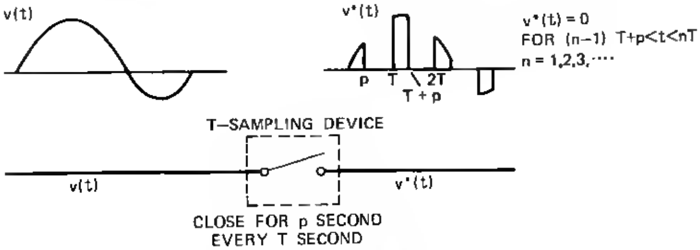


Fig. 1—Sampling device.

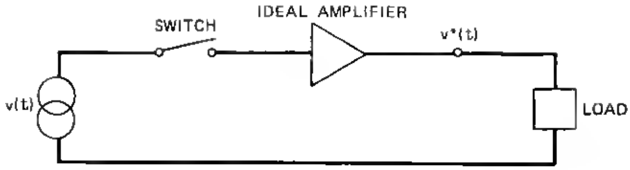


Fig 2—Time-division switch followed by an ideal amplifier.

the sampled-data system techniques are directly applicahle. In practice, this is not always the case. Sampling in a time-division system is frequently performed hy a switch connected directly to a time-division hus. Our ohjective is to characterize the switch hy a sampler plus some modified system-transfer function so that any time-division system employing periodic sampling can he treated as a standard sampled-data system.

In general, the time-division system we are interested in has the form shown in Fig. 3. It consists of two networks connected hy a switch in series with some finite impedance Z_o. The switch is closed periodically for a hrief interval of p seconds every T seconds. The smallest time constant of the input signal and the sampling period T are both much greater than p. Referring to Fig. 3, we define $v_{12}(t)$ as the difference hetween $v_1(t)$ and $v_2(t)$, the voltages at terminals 1 and 2, respectively; $v_{oc}(t)$ is defined as the open circuit Thevenin equivalent voltage at terminal 1 and $i(t)$ as the current in the switch. The current $i(t)$ can he found from the equivalent circuit (Fig. 4) ohtained hy connecting the driving source $e(t) = v_{oc}(t)$ in series with the time-division switch and impedances Z_0, Z_1, and Z_2, where Z_1 and Z_2 are the output impedance of network 1 and input impedance of network 2, respectively. As the switch is closed only for a time interval from $t = nT$ to

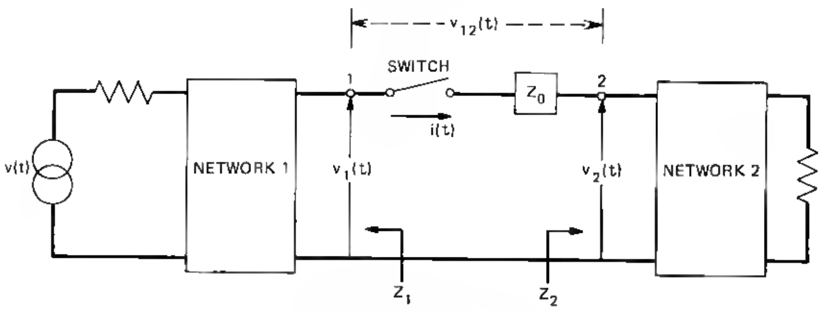


Fig. 3—General time-division system.

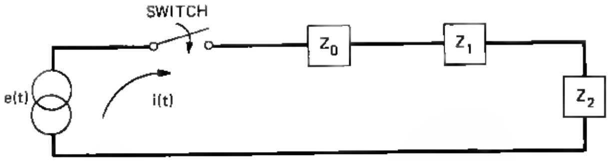


Fig. 4—Equivalent circuit for solving for $g(t)$.

$t = nT + p$, $n = 0, 1, 2, \cdots$, we may express $i(t)$ as

$$i(t) = \begin{cases} i_n(t) & nT \leqq t \leqq nT + p \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

or

$$i(t) = \sum_{n=0}^{\infty} i_n(t) \cdot [u(t - nT) - u(t - nT - p)], \tag{3}$$

where $u(t)$ is the unit step-function, T is the sampling period, and p is the sampling duration.

To solve for $i_n(t)$, we let the switcb in the equivalent circuit (Fig. 4) close only for a time interval from $t = nT$ to $t = nT + p$. The driving source $e(t)$ should be modified to $v_{oc}(t) - v_{10}(nT^-) - v_{20}(nT^-)$, where

$$\begin{aligned} v_{10}(t) &= v_{oc}(t) - v_1(t) = i(t) \circ z_1(t) \\ v_{20}(t) &= v_2(t) = i(t) \circ z_2(t) \end{aligned} \tag{4}$$

and $\circ$ denotes the convolution product. Note that $v_{10}(nT^-)$ and $v_{20}(nT^-)$ are tbe voltages across Z_1 and Z_2 just before the switch is closed. For the small time interval $nT < t < nT + p$, $v_{oc}(t) \simeq v_{oc}(nT)$.[10] Therefore, $e(t) \simeq v_{oc}(nT) - v_{10}(nT^-) - v_{20}(nT^-)$. Defining*

$$\begin{aligned} v_d(t) &= v_{oc}(t) - v_{10}(t^-) - v_{20}(t^-) \\ &= v_{oc}(t) - v_{10}(t - \epsilon) - v_{20}(t - \epsilon), \end{aligned} \tag{5}$$

where $\epsilon > 0$ is an arbitrarily small number, tbe current ${}_ni(t)$ can be expressed as

$$i_n(t) = \mathcal{L}^{-1}\left[\frac{v_d(nT)}{S} \cdot Y(S) \cdot e^{-nTS}\right], \tag{6}$$

where $Y(S) = 1/[Z_0(S) + Z_1(S) + Z_2(S)]$ is tbe admittance function of the equivalent circuit and $\mathcal{L}^{-1}$ denotes inverse Laplace trans-

* From (5) we note that at the sample instant $t = (nT)$, $v_d(nT) = v_1(nT^-) - v_2(nT^-)$ for $n \geqq 1$, and $v_d(0) = v_{oc}(0) = 0$ for a physically realizable system. Hence, $v_d(nT) = v_{12}(nT^-)$, the difference between $v_1(t)$ and $v_2(t)$ just before the switch is closed.

formation. Substituting (6) into (3), we have

$$i(t) = \sum_{n=0}^{\infty} v_d(nT)\mathcal{L}^{-1}\left[\frac{1}{S}\cdot Y(S)\cdot e^{-nTS}\right] \cdot [u(t - nT) - u(t - nT - p)]. \quad (7)$$

Equation (7) suggests that we may characterize the switch by a sampler plus a transfer function $G(S)$, as shown in Fig. 5, to relate v_d and i by

$$I(S) = V_d^*(S)G(S), \quad (8)$$

where $I(S)$, $V_d^*(S)$, and $G(S)$ are the Laplace transforms of $i(t)$, $v_d^*(t)$, and $g(t)$, respectively (similar notations will be used hereafter without explanation). By the definition of impulse-modulated signal,

$$v_d^*(t) = \sum_{n=0}^{\infty} v_d(nT)\delta(t - nT). \quad (9)$$

In the time domain, eq. (8) corresponds to the convolution integral

$$i(t) = \int_0^t v_d^*(\tau)g(t - \tau)d\tau. \quad (10)$$

Substituting (9) into (10) and integrating we have

$$i(t) = \sum_{n=0}^{\infty} v_d(nT)g(t - nT)u(t - nT). \quad (11)$$

Comparing (7) and (11), we have

$$g(t)u(t) = \mathcal{L}^{-1}\left[\frac{1}{S}\cdot Y(S)\right]\cdot[u(t) - u(t - p)] \quad (12)$$

and

$$G(S) = \mathcal{L}\left\{\mathcal{L}^{-1}\left[\frac{1}{S}\cdot Y(S)\right]\cdot[u(t) - u(t - p)]\right\}, \quad (13)$$

where $\mathcal{L}$ is the Laplace transformation operator. Equation (13) yields the transfer function we need to characterize the switch. Note that the function $\mathcal{L}^{-1}[(1/S)Y(s)]$ is the current $i(t)$ in Fig. 4 with a unit dc driving source. Once $G(S)$ is found, a functional block diagram

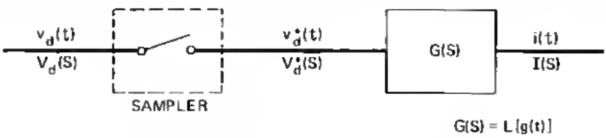


Fig. 5—Characterization of the switch.

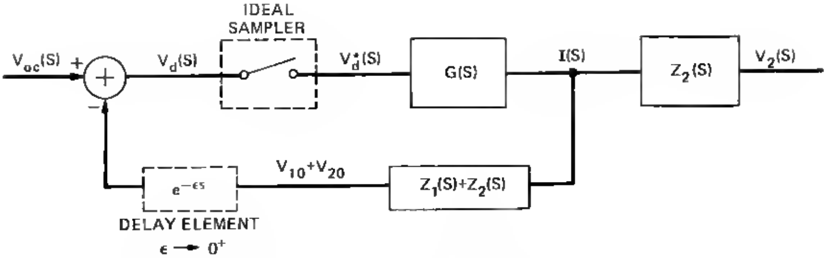


Fig. 6—Transfer function block diagram between V_{oc} and V_2.

describing the signal flow from network 1 to network 2 can be constructed easily, as shown in Fig. 6. Now the system in Fig. 3 is converted to a standard sampled-data system.

The transfer function from v_{oc} to v_2 can be obtained from Fig. 6:

$$\begin{aligned} V_d(S) &= V_{oc}(S) - V_{10}(S)e^{-\epsilon S} - V_{20}(S)e^{-\epsilon S} \\ &= V_{oc}(S) - I(S)[Z_1(S) + Z_2(S)]e^{-\epsilon S} \\ &= V_{oc}(S) - V_d^*(S)G(S)[Z_1(S) + Z_2(S)]e^{-\epsilon S} \\ V_d^*(S) &= V_{oc}^*(S) - V_d^*(S)[GZ_1(S)e^{-\epsilon S} + GZ_2(S)e^{-\epsilon S}]^* \end{aligned}$$

or

$$V_d^*(S) = \frac{V_{oc}^*(S)}{1 + [GZ_1(S)e^{-\epsilon S} + GZ_2 e^{-\epsilon S}]^*}, \tag{14}$$

where $GZ_i(S) = G(S)Z_i(S)$ and $i = 1, 2$, and $\epsilon > 0$ is arbitrarily small. From (14),

$$\begin{aligned} V_2(S) &= I(S)Z_2(S) = V_d^*(S)G(S)Z_2(S) \\ &= \frac{GZ_2(S)}{1 + [GZ_1(S)e^{-\epsilon S} + GZ_2(S)e^{-\epsilon S}]^*} V_{oc}^*(S). \end{aligned} \tag{15}$$

To find $[GZ_i(S)e^{-\epsilon S}]^*$, we need to know the relationship between $gz_i^*(t)$ and $[gz_i(t - \epsilon)]^*$. The function $g(t)$ is defined in (12):

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{1}{S[Z_0(S) + Z_1(S) + Z_2(S)]}\right\}[u(t) - u(t-p)] \\ &= h(t)u(t) - h(t)u(t-p), \end{aligned}$$

where

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{S[Z_0(S) + Z_1(S) + Z_2(S)]}\right\}.$$

Note that $h(t)$ is the step-response of a linear passive network and, thus,

is continuous for all $t > 0$. Now

$$gz_i(t) = \int_{-\infty}^{t} g(\tau)z_i(t-\tau)d\tau$$

$$= \int_{-\infty}^{t} h(\tau)z_i(t-\tau)u(\tau)d\tau - \int_{-\infty}^{t} h(\tau)z_i(t-\tau)u(\tau-p)d\tau.$$

From the convolution of two continuous time functions, $gz_1(t)$ will be continuous for all $0 < t < p$ and $t > p$. Since $p < T$, $gz_i(t)$ will be continuous at $t = nT$ for all $n \geqq 1$, i.e.,

$$gz_i(nT - \epsilon) = gz_i(nT) \tag{16}$$

as ϵ approaches zero. At $t = 0$, since $gz_i(t) = 0$ for all $t < 0$, we have

$$gz_i(0 - \epsilon) \triangleq 0. \tag{17}$$

From (16) and (17),

$$\sum_{n=0}^{\infty} gz_i(nT - \epsilon)\delta(t - nT) = \sum_{n=1}^{\infty} gz_i(nT)\delta(t - nT)$$

$$= \sum_{n=0}^{\infty} gz_i(nT)\delta(t - nT) - gz_i(0)\delta(t)$$

for arbitrarily small ϵ. Therefore,

$$[gz_i(t - \epsilon)]^* = gz_i^*(t) - gz_i(0)\delta(t) \tag{18}$$

and

$$[GZ_i(S)e^{-\epsilon S}]^* = GZ_i^*(S) - gz_i(0). \tag{19}$$

From (15) and (19), we have:

$$\frac{V_2(S)}{V_{oc}^*(S)} = \frac{GZ_2(S)}{1 - gz_1(0) - gz_2(0) + GZ_1^*(S) + GZ_2^*(S)} \tag{20}$$

and

$$\frac{V_2^*(S)}{V_{oc}^*(S)} = \frac{GZ_2^*(S)}{1 - gz_1(0) - gz_2(0) + GZ_1^*(S) + GZ_2^*(S)}. \tag{21}$$

If we are interested in the transfer function $V_2(S)/V_1^*(S)$, then since $V_1(S) = V_{oc}(S) - I(S)Z_1(S)$, we have:

$$V_1^*(S) = V_{oc}^*(S) - V_d^*(S)GZ_1^*(S). \tag{22}$$

Substituting (14) into (22),

$$V_1^*(S) = V_{oc}^*(S)\,\frac{1 + GZ_2^*(S) - gz_1(0) - gz_2(0)}{1 + GZ_1^*(S) + GZ_2^*(S) - gz_1(0) - gz_2(0)}. \tag{23}$$

From (20), (21), and (23), we have:

$$\frac{V_2(S)}{V_1^*(S)} = \frac{GZ_2(S)}{1 - gz_1(0) - gz_2(0) + GZ_2^*(S)} \tag{24}$$

and

$$\frac{V_2^*(S)}{V_1^*(S)} = \frac{GZ_2^*(S)}{1 - gz_1(0) - gz_2(0) + GZ_2^*(S)}. \tag{25}$$

Since $gz_i(0) = \lim_{S\to\infty} S\cdot GZ_i(S)$, it will be equal to zero when the function $GZ_i(S)$ has at least two more poles then zeros. In such cases, (20), (21), (24), and (25) become

$$\frac{V_2(S)}{V_{oc}^*(S)} = \frac{GZ_2(S)}{1 + GZ_1^*(S) + GZ_2^*(S)} \tag{20A}$$

$$\frac{V_2^*(S)}{V_{oc}^*(S)} = \frac{GZ_2^*(S)}{1 + GZ_1^*(S) + GZ_2^*(S)} \tag{21A}$$

$$\frac{V_2(S)}{V_1^*(S)} = \frac{GZ_2(S)}{1 + GZ_2^*(S)} \tag{24A}$$

$$\frac{V_2^*(S)}{V_1^*(S)} = \frac{GZ_2^*(S)}{1 + GZ_2^*(S)}. \tag{25A}$$

As a simple example, let us refer to Fig. 7, which shows an ideal sample-and-hold circuit with a capacitor C. For this circuit $Z_0 = Z_1 = 0$, $Z_2 = 1/CS$, and it can easily be found that $G(S) = C$. Therefore,

$$G(S)Z_2(S) = \frac{1}{S}. \tag{26}$$

Since it has only one more pole than it has zeros, eq. (24) should be used to find $V_2(S)/V_1^*(S)$:

$$\frac{V_2(S)}{V_1^*(S)} = \frac{1/S}{[1 - u(0)] + [1/S]^*} = \frac{1 - e^{-TS}}{S}. \tag{27}$$

The general approach used to characterize the time-division switch can be summarized as follows:

(*i*) Form an equivalent circuit (Fig. 4) by using a unit step voltage source to drive the impedances Z_0, Z_1, and Z_2 connected in series, where Z_1 is the output impedance of $N1$ and Z_2 is the input impedance of $N2$.

(*ii*) Solve for the current $i(t)$ in the equivalent circuit.

(*iii*) Let $g(t) = i(t)[u(t) - u(t - p)]$. Calculate $G(S) = \mathfrak{L}\{g(t)\}$ and $GZ_i(S) = G(S)Z_i(S)$, $i = 1, 2$.

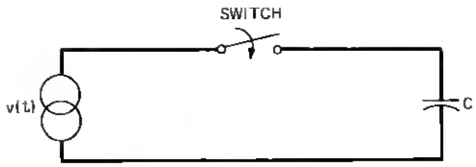


Fig. 7—Ideal sample-and-hold circuit.

(*iv*) Now the energy transfer between $N1$ and $N2$ can be described by a sampler plus some transfer functions. Either one of the following formulas may be used:

$$\frac{V_2(S)}{V_{oc}^*(S)} = \frac{GZ_2(S)}{1 - gz_1(0) - gz_2(0) + GZ_1^*(S) + GZ_2^*(S)} \tag{28}$$

$$\frac{V_2(S)}{V_1^*(S)} = \frac{GZ_2(S)}{1 - gz_1(0) - gz_2(0) + GZ_2^*(S)}, \tag{29}$$

where v_{oc} is the open circuit voltage at terminal 1 in Fig. 3. Wben the function $GZ_i(S)$ bas at least two more poles than zeros, we know immediately that $gz_i(0) = 0$ and can be removed from tbe above formulas.

III. AN APPLICATION

The switch we model here is a practically realizable sample-and-hold switch for a time-division switching system. It is shown in Fig. 8. Tbe series resistor R represents the gate resistance during sampling. Tbe series inductor represents the lead inductance whose value depends on the bus structure.

For this circuit, $Z_1 = 0$, $Z_2 = 1/CS$, and $Z_0 = SL + R$. Tberefore, $g(t)$ can be found by solving a simple series RLC circuit witb unit dc input and a switch closed at $t = 0$ and open at $t = p$ for the rest of tbe time. The result is

$$g(t) = \frac{1}{2L\sqrt{\alpha^2 - \omega_0^2}} (e^{-\beta_1 t} - e^{-\beta_2 t})[u(t) - u(t - p)], \tag{30}$$

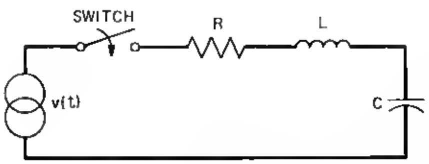


Fig. 8—Practical sample-and-hold switch.

where $\alpha = R/2L$, $\omega_0 = 1/\sqrt{LC}$, $\beta_1 = \alpha - \sqrt{\alpha^2 - \omega_0^2}$, and $\beta_2 = \alpha + \sqrt{\alpha^2 - \omega_0^2}$. From (30) we have:

$$G(S) = \frac{1}{2L\sqrt{\alpha^2 - \omega_0^2}} \left\{ \frac{1 - e^{-p(S+\beta_1)}}{S + \beta_1} - \frac{1 - e^{-p(S+\beta_2)}}{S + \beta_2} \right\}. \tag{31}$$

The transfer function $H(S) = V_2(S)/V_1^*(S)$ can now he found from

$$H(S) = \frac{V_2(S)}{V_1^*(S)} = \frac{G(S)Z_2(S)}{1 + [G(S)Z_2(S)]^*}. \tag{32}$$

The calculation of $G(S)Z_2(S)$ and $[G(S)Z_2(S)]^*$ is given in the appendix. With $G(S)Z_2(S)$ and $[G(S)Z_2(S)]^*$ known, we have

$$H(S) = \frac{1 - e^{-TS}}{S} \cdot \frac{\beta_1\beta_2}{\beta_1 - \beta_2} \left[\frac{1 - e^{-p(S+\beta_1)}}{S + \beta_1} - \frac{1 - e^{-p(S+\beta_2)}}{S + \beta_2} \right] \cdot \frac{1}{1 + ke^{-TS}}, \tag{33}$$

where

$$k = \frac{\beta_1 e^{-\beta_2 p} - \beta_2 e^{-\beta_1 p}}{\beta_2 - \beta_1}. \tag{34}$$

From (33),

$$H^*(S) = \frac{V_2^*(S)}{V_1^*(S)} = \frac{1}{1 + \dfrac{1 - e^{-TS}}{e^{-TS}} \cdot \dfrac{1}{1 + k}} = \frac{1 + k}{e^{TS} + k}. \tag{35}$$

When the driving function is sinusoidal, we have:

$$H^*(j\omega) = \frac{1 + k}{e^{j\omega T} + k} \tag{36}$$

$$|H^*(j\omega)| = \frac{1 + k}{\sqrt{1 + k^2 + 2k \cos \omega T}}. \tag{37}$$

This shows that, for $k \neq 0$, the magnitude of the voltage gain at the sampling instant will he a function of frequency. The maximum (or minimum, depending upon the sign of k*) occurs at $\omega T = \pi$, i.e., the

* It can be easily verified from (34) that k will be positive only if the equivalent RLC circuit is under damped.

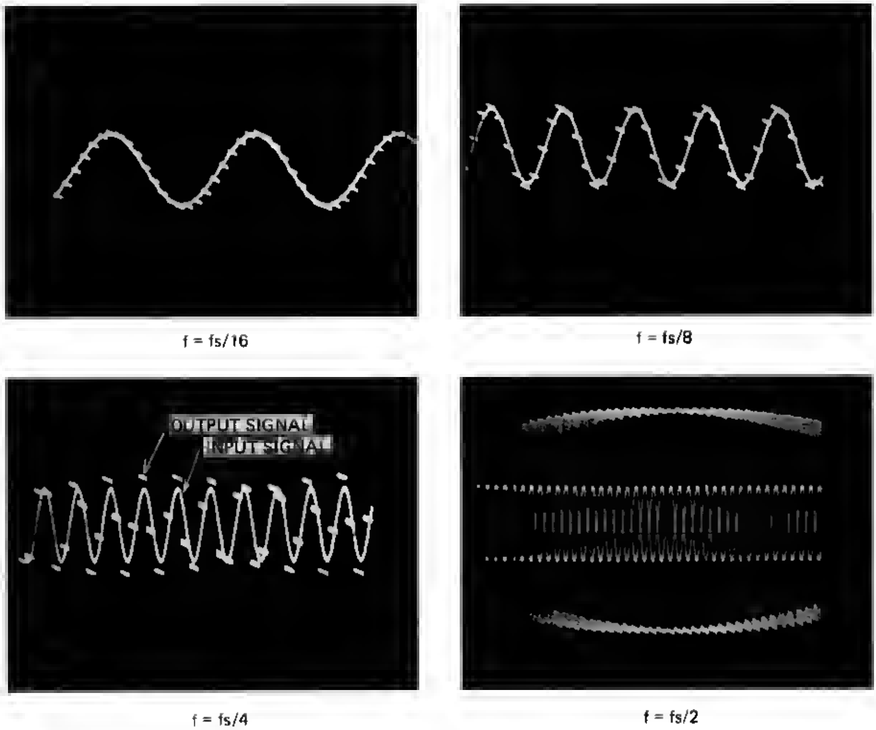


Fig. 9—Variation of magnitude of output signal at sampling gate with respect to input frequency f.

half sampling frequency:

$$|H^*(j\omega)|_{\text{extreme}} = \frac{1 + k}{|1 - k|}. \tag{38}$$

Figure 9 shows the laboratory observation of such an effect for $k \simeq \frac{1}{3}$.

When the lead inductance is negligibly small, i.e., $L \to 0$, then $\beta_2 \to \infty$ and $\beta_1 \to 1/RC$. Equations (33), (35), and (37) now become:

$$H(S) = \frac{1 - e^{-TS}}{S} \frac{a}{S + a} \frac{1 - e^{-ap}e^{-pS}}{1 - e^{-ap}e^{-TS}} \tag{39}$$

$$H^*(S) = \frac{1 - e^{-ap}}{e^{TS} - e^{-ap}} \tag{40}$$

$$|H^*(j\omega)| = \frac{1 - e^{-ap}}{\sqrt{1 + e^{-2ap} - 2e^{-ap}\cos\omega T}}, \tag{41}$$

where $a = 1/RC$.

In this case, the RLC-series circuit reduces to an RC circuit. In a practical time-division-switching system, the typical values might be: sampling period $T = 83.3\ \mu s$ (sampling at 12 kHz), sampling duration $p = 300$ ns, hold-capacitor $C = 1000$ pF, and gate resistance $R = 30\ \Omega$. A simple calculation will show that

$$ap = \frac{p}{RC} = \frac{300 \times 10^{-9}}{30 \times 10^{-9}} = 10. \tag{42}$$

Hence, $e^{-ap} = e^{-10} \simeq 0$ and the transfer function $H(S)$ in (39) becomes

$$H(S) \simeq \frac{1 - e^{-TS}}{S} \cdot \frac{\frac{1}{RC}}{S + \frac{1}{RC}}, \tag{43}$$

which indicates that the switch-and-hold circuit can be approximately considered as an ideal sample-and-hold device in series with an RC circuit, as shown in Fig. 10.

It is also interesting to note that if both the gate resistance R and the lead inductance L approach 0, we will have an ideal sample-and-hold switch. From (43), we can see that $H(S)$ will approach the ideal sample-and-hold transfer function $1 - e^{-TS}/S$ as expected.

IV. AN APPROXIMATION

In this section, we shall present a simplified approach which in general leads to a very good approximation of the results found by the general approach described in Section III. The basic idea here is to approximate the current $i(t)$ in the switch by an impulse-modulated signal,

$$i(t) \simeq \bar{\imath}(t), \tag{44}$$

and characterize the switch by the energy transfer during the sampling duration:

$$v_2(nT^+) = v_2(nT^-) + \gamma[v_1(nT^-) - v_2(nT^-)], \tag{45}$$

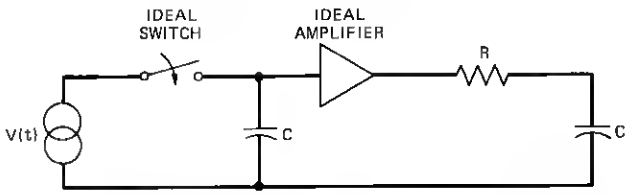


Fig. 10—Equivalent circuit for Fig. 8 when L approaches 0.

where nT^- represents the instant just before the switch is closed, and nT^+ the instant just after the switch is reopened. The determination of γ, which is related to the transfer loss, will be discussed later. From (45) we have:

$$v_2^*(t) = \gamma[v_1(t-\epsilon)]^* + (1-\gamma)[v_2(t-\epsilon)]^* \tag{46}$$

and†

$$V_2^*(S) = \gamma[V_1(S)e^{-\epsilon S}]^* + (1-\gamma)[V_2(S)e^{-\epsilon S}]^*, \tag{47}$$

where ϵ is arbitrarily small.

Substituting

$$V_1(S) = V_{oc}(S) - \tilde{I}^*(S)Z_1(S)$$

and

$$V_2(S) = \tilde{I}^*(S)Z_2(S)$$

into (47), we have

$$\tilde{I}^*(S)Z_2^*(S) = \gamma[V_{oc}(S)e^{-\epsilon S}]^* + \tilde{I}^*(S)\{-\gamma[Z_1(S)e^{-\epsilon S}]^* + (1-\gamma)[Z_2(S)e^{-\epsilon S}]^*\}. \tag{48}$$

As $V_{oc}(t)$ is continuous for all $t \geqq 0$, and $z_i(t)$ is continuous for all $t > 0$,

$$\tilde{I}^*(S)Z_2^*(S) = \gamma V_{oc}^*(S) + \tilde{I}^*(S)\{-\gamma[Z_1^*(S) - z_1(0)] + (1-\gamma)[Z_2^*(S) - z_2(0)]\}$$

or

$$\tilde{I}^*(S) = \frac{\gamma V_{oc}^*(S)}{\gamma[Z_1^*(S) + Z_2^*(S)] - \gamma z_1(0) - (\gamma - 1)z_2(0)}. \tag{49}$$

Therefore,

$$\frac{V_2(S)}{V_{oc}^*(S)} = \frac{\tilde{I}^*(S)Z_2(S)}{V_{oc}^*(S)} = \frac{\gamma Z_2(S)}{\gamma[Z_1^*(S) + Z_2^*(S)] - \gamma z_1(0) - (\gamma - 1)z_2(0)}, \tag{50}$$

and

$$\frac{V_2(S)}{V_1^*(S)} = \frac{\tilde{I}^*(S)Z_2(S)}{V_{oc}^*(S) - \tilde{I}^*(S)Z_1^*(S)} = \frac{\gamma Z_2(S)}{\gamma Z_2^*(S) - \gamma z_1(0) - (\gamma - 1)z_2(0)}. \tag{51}$$

† Note that as $v_i(t)$ may not be continuous at $t = nT$, $n = 0, 1, \cdots$,

$$[V_i(S)e^{-\epsilon S}]^* \neq V_i^*(S) - v_i(0).$$

To determine the constant γ, we note that the change in v_2 during the sampling duration is a function of the current in the sampling switch:

$$\begin{aligned}\Delta v_2(nT) &= v_2(nT^+) - v_2(nT^-) \\ &= i_n(t) \cdot z_2(t) \,|_{t=nT+p}.\end{aligned} \tag{52}$$

As mentioned earlier, $i_n(t)$ can be solved from Fig. 4 with the driving source $e(t) = v_d(nT)$ and the switch closed at $t = nT$. Since $v_{oc}(t)$ is continuous for all $t \geqq 0$, $v_d(nT) = v_{12}(nT^-) = v_1(nT^-) - v_2(nT^-)$. Hence, in the equivalent circuit of Fig. 4, we let $e(t) = v_{12}(nT^-)$ and close the switch at $t = nT$. Solving this circuit, the current will be $i_n(t)$ and the voltage across Z_2 at $t = nT + p$ will be $\Delta v_2(nT)$; i.e.,

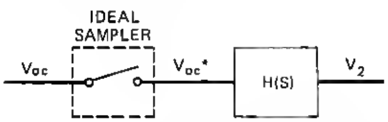
$$\Delta v_2(nT) = \mathcal{L}^{-1}\left\{\mathcal{L}\left[\frac{v_{12}(nT^-)}{S}\cdot Y(S)\cdot e^{-nTS}\cdot Z_2(S)\right]\right\}_{t=nT+p}, \tag{53}$$

where $Y(S) = 1/[Z_0(S) + Z_1(S) + Z_2(S)]$. From (53),

$$\begin{aligned}\gamma &= \frac{\Delta v_2(nT)}{v_{12}(nT^-)} \\ &= \mathcal{L}^{-1}\left\{\mathcal{L}\left[\frac{1}{S}\cdot Y(S)\cdot e^{-nTS}\cdot Z_2(S)\right]\right\}_{t=nT+p} \\ &= \mathcal{L}^{-1}\left\{\mathcal{L}\left[\frac{1}{S}\cdot Y(S)\cdot Z_2(S)\right]\right\}_{t=p}.\end{aligned} \tag{54}$$

Hence, in Fig. 4, if we let $e(t) = u(t)$ and close the switch at $t = 0$, then the voltage across Z_2 at $t = p$ will be the value of γ. After γ is found, either (50) or (51) may be used to enable us to replace the switch by an ideal sampler plus a transfer function, as shown in Fig. 11.

To illustrate how this approach works, let us return to the practical sample-and-hold switch in Fig. 8. As stated in the last section, $Z_1 = 0$, $Z_0 = R + SL$, and $Z_2 = 1/CS$. Solving the series RLC circuit with the



$$H(S) = \frac{\gamma Z_2(S)}{\gamma\left[Z_1^*(S)+Z_2^*(S)\right] - \gamma z_1(0) - (\gamma-1)\, z_2(0)}$$

Fig. 11—Transfer function diagram for the approximate approach.

driving source of $u(t)$, the current $i_0(t)$ is found as shown in (30):

$$i_0(t) = \frac{1}{2L\sqrt{\alpha^2 - \omega_0^2}} (e^{-\beta_1 t} - e^{-\beta_2 t}). \tag{55}$$

Now γ, the voltage across Z_2 at $t = p$, can be found:

$$\begin{aligned}\gamma &= \frac{1}{C}\int_0^p i_0(t)dt \\ &= \frac{\beta_1\beta_2}{\beta_2 - \beta_1}\left[\frac{1}{\beta_1}(1 - e^{-\beta_1 p}) - \frac{1}{\beta_2}(1 - e^{-\beta_2 p})\right] \\ &= 1 + k,\end{aligned} \tag{56}$$

where

$$k = \frac{\beta_1 e^{-\beta_2 p} - \beta_2 e^{-\beta_1 p}}{\beta_2 - \beta_1}$$

is the same k given hy (34) in the last section. Therefore,

$$\begin{aligned}\tilde{H}(S) &= \frac{V_2(S)}{V_1^*(S)} = \frac{(1 + k)\dfrac{1}{CS}}{(1 + k)\left[\dfrac{1}{CS}\right]^* - \dfrac{k}{C}} \\ &= \frac{1 - e^{-TS}}{S}\,\frac{1 + k}{1 + ke^{-TS}}.\end{aligned} \tag{57}$$

We now want to show that $\tilde{H}(S)$ of (57) is a good approximation of $H(S)$ of (33). From (33), we have:

$$H(S) = \frac{1 - e^{-TS}}{S}\,\frac{F(S)}{1 + ke^{-TS}}, \tag{58}$$

where

$$F(S) = \frac{\beta_1\beta_2}{\beta_2 - \beta_1}\left[\frac{1 - e^{-p(S+\beta_1)}}{S + \beta_1} - \frac{1 - e^{-p(S+\beta_2)}}{S + \beta_2}\right]. \tag{59}$$

To show that $H(S) \simeq \tilde{H}(S)$, we want to show that $F(S) \simeq 1 + k$. Since p is small, we have:

$$\begin{aligned}F(S) &\simeq \frac{\beta_1\beta_2}{\beta_2 - \beta_1}\left[p - \frac{p^2}{2}(S + \beta_1) - p + \frac{p^2}{2}(S + \beta_2)\right] \\ &= \frac{\beta_1\beta_2}{2}p^2,\end{aligned}$$

and

$$1 + k = 1 + \frac{\beta_1 e^{-\beta_2 p} - \beta_2 e^{-\beta_1 p}}{\beta_2 - \beta_1} = \frac{\beta_2(1 - e^{-\beta_1 p}) - \beta_1(1 - e^{-\beta_2 p})}{\beta_2 - \beta_1}$$

$$\simeq \frac{1}{\beta_2 - \beta_1}\left[\beta_2\left[\beta_1 p - \frac{\beta_1^2 p^2}{2}\right] - \beta_1\left[\beta_2 p - \frac{\beta_2^2 p^2}{2}\right]\right]$$

$$= \frac{\beta_1\beta_2}{2}p^2.$$

Hence, $F(S) \simeq 1 + k$ and $H(S) \simeq \tilde{H}(S)$.

Finally, we note that as Z_0 approaches zero, the current $i(t)$ does approach an impulse-modulated function. Thus, the approach described in this section will always lead to the true answer when $Z_0 = 0$. For example, when R, $L \to 0$, the switch we modeled above becomes the ideal sample-and-hold switch. In this case $k \to 0$ and $\tilde{H}(S)$ of (57) approaches $1 - e^{-TS}/S$, the familiar ideal sample-and-hold transfer function. It can also be easily seen that if we start with this ideal sample-and-hold switch, i.e., $Z_0 = Z_1 = 0$ and $Z_2 = 1/CS$, then $\gamma = 1$ and $Z_2^*(S) = 1/C(1 - e^{-TS})$. From (51), we shall again have $V_2(S)/V_1^*(S) = 1 - e^{-TS}/S$ as expected.

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APPENDIX

*Calculation of G(S)Z₂(S) and [G(S)Z₂(S)]**

Since $Z_2(S) = 1/CS$, $\beta_1 = \alpha - \sqrt{\alpha^2 - \omega_0^2}$, $\beta_2 = \alpha + \sqrt{\alpha^2 - \omega_0^2}$, and $\omega_0^2 = 1/LC$, we have:

$$GZ(S) = G(S)Z_2(S) = \frac{1}{LC}\frac{1}{2\sqrt{\alpha^2 - \omega_0^2}}\cdot\frac{1}{S}\cdot\left\{\frac{1 - e^{-p(S+\beta_1)}}{S + \beta_1} - \frac{1 - e^{-p(S+\beta_2)}}{S + \beta_2}\right\}$$

$$= \frac{\beta_1\beta_2}{\beta_2 - \beta_1}\cdot\frac{1}{S}\left\{\frac{1 - e^{-p(S+\beta_1)}}{S + \beta_1} - \frac{1 - e^{-p(S+\beta_2)}}{S + \beta_2}\right\} \tag{60}$$

and

$$GZ^*(S) = \frac{\beta_1\beta_2}{\beta_2 - \beta_1}\left\{\sum_{i=1}^{2}\frac{(-1)^{i+1}}{\beta_i} \times\left[\frac{1}{S} - \frac{e^{-p(S+\beta_i)}}{S} - \frac{1}{S + \beta_i} + \frac{e^{-p(S+\beta_i)}}{S + \beta_i}\right]^*\right\}.$$

Hence,

$$
\begin{aligned}
GZ^*(S) &= \frac{\beta_1\beta_2}{\beta_2-\beta_1}\left\{\sum_{i=1}^{2}\frac{(-1)^{i+1}}{\beta_i}\left[\frac{1}{1-e^{-TS}}-\frac{e^{-\beta_i p}e^{-TS}}{1-e^{-TS}}\right.\right.\\
&\qquad\left.\left.-\frac{1}{1-e^{-TS}e^{-\beta_i T}}+\frac{e^{-\beta_i T}e^{-TS}}{1-e^{-TS}e^{-\beta_i T}}\right]\right\}\\
&= \frac{\beta_1\beta_2}{\beta_2-\beta_1}\sum_{i=1}^{2}\frac{(-1)^{i+1}}{\beta_i}\left[\frac{1-e^{-\beta_i p}e^{-TS}}{1-e^{-TS}}-1\right]\\
&= \frac{\beta_1\beta_2}{\beta_2-\beta_1}\cdot\frac{e^{-TS}}{1-e^{-TS}}\left[\frac{1-e^{-\beta_1 p}}{\beta_1}-\frac{1-e^{-\beta_2 p}}{\beta_2}\right]\\
&= \frac{1}{\beta_2-\beta_1}\cdot\frac{e^{-TS}}{1-e^{-TS}}\left[\beta_2-\beta_1+\beta_1 e^{-\beta_2 p}-\beta_2 e^{-\beta_1 p}\right]\\
&= (1+k)\,\frac{e^{-TS}}{1-e^{-TS}}, \qquad (61)
\end{aligned}
$$

where

$$k = \frac{\beta_1 e^{-\beta_2 p}-\beta_2 e^{-\beta_1 p}}{\beta_2-\beta_1}. \qquad (62)$$

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